Propagation from a Point Source in a Randomly Refracting Medium

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This paper considers the propagation of scalar (acoustic) waves from a single-frequency point source imbedded in a medium with random refractive index, in contrast with the usual plane-wave case in which the source is far removed from the medium. With the index being a statistically homogeneous and isotropic function of position, but not a function of time, the average complex field $u_o(r) = \langle u(r) \rangle$ *and the spatial covariance* $\langle u_i(r)u_i^*(\rho) \rangle$ *of the fluctuation field* $u_i(r) = u(r) - u_o(r)$ *are calculated. Beyond a few correlation lengths from the source, the average field can be approximated by a spherical wave with the same complex wavenumber found in the plane-wave case. A near-source wave number is also obtained. Under an improved far-field condition, the spatial covariance is reduced to spectral integration formulas for both transverse and longitudinal separation of the receiving points. These formulas reveal that correlation lengths are much longer in the point-source case than in the plane-wave case, even though the relative variances are the same. We illustrate this result with plots for an exponential index spectrum and for a constant spectrum.*

I. INTRODUCTION

For analysis of a detection or communication system which processes signals from an array of sensors, a convenient postulate is that the signal field in the vicinity of the array is a plane wave (or perhaps a finite collection of plane waves in the multipath case). Under such a postulate, coherent addition of the sensor outputs can yield array gain and directivity in the presence of ambient noise. However, there is always some disparity between the predicted performance and the performance realized in practice. In part, the disparity can be attributed to shortcomings in the signal model, the field not being a time-invariant plane wave in the vicinity of the array. The output of a single sensor may not be constant in time but instead is apt to

fade. Moreover, the outputs of different sensors do not fade "in step"; that is, after the array is steered, the signals do not fade with the unity correlation predicted by a fading-plane-wave model. Instead, the signals fade with correlation less than unity. The origin of these fading phenomena is the subject of this paper.

A simplified model of fading is considered within the framework of the following assumptions. For a short period of time, the transmission properties of a propagation path are constant; then they undergo small deviations to attain another constant configuration for the next short period of time. These short-term deviations are relative to some nominal or average configuration, as opposed to representing a slow gross trend of the overall path properties. Such short-term deviations are modeled here by the effects of random fluctuations of the index of refraction, which could be associated in the underwater acoustic case, for example, with the temperature microstructure, turbulence, and circulatory motion of water masses. Deviations of path properties associated with fluctuations of a surface of reflection are not incorporated into the model. Thus, the model is most appropriate for short-term deviations of the properties of a pure-refracted path.

In the specific situation analyzed below, the acoustic source is a single-frequency point source suspended far from any boundaries. If the refractive index were nonrandom and not position dependent, the acoustic field would be the usual spherical wave. Instead, the refractive index is a random function of position, but not of time. The average value of the index is not position dependent, so that the average line-of-sight ray path is straight rather than bent. The spatial covariance of the index is a function of the magnitude of the position-difference vector (the index is second-order homogeneous and isotropic). The problem is to find the average and spatial covariance of the acoustic field.

Much of the literature (for example, Refs. 1–3 and most of Ref. 4) is concerned not with the above spherical-wave problem but with a situation in which plane waves impinge upon a half-space with random refractive index. One essential difference is that the spherical-wave source is imbedded in the random medium whereas the plane-wave source is far removed from the random medium. Regardless of how large the distance from the spherical-wave source to an observation point becomes, this difference of configuration is preserved.†

† The configurations are called the "radio link problem" (spherical) and the "radio star problem" (planar) in Ref. 9.

Some aspects of the spherical-wave case have been treated with the Rytov method (Refs. 4–5) and other techniques (Refs. 6–8).

This analysis treats the spherical-wave prohlem with a version of perturhation theory previously applied to the plane-wave prohlem.[2,3] For distances greater than a few correlation lengths, the average field can be approximated hy a spherical wave with the same complex-valued wave number previously derived in the plane-wave case.[2] A near-source wave numher is also ohtained. On the other hand, it is found that the covariance function of the fluctuation field exhihits much larger correlation lengths in the spherical-wave case than in the plane-wave case.[3,4] This conclusion follows from simple integration formulas for the covariances and is illustrated hy plots of the covariance for special cases.

II. PERTURBATION THEORY

We consider the propagation of acoustic waves in a random time-invariant medium for the case of a monochromatic omnidirectional source. Our model is the Helmholtz equation:

$$[\nabla^2 + (1 + \mu(r))^2 k_o^2]u(r) = -\delta(r) \tag{1}$$

where ∇^2 is the Laplacian $\nabla \cdot \nabla$, $\mu(r)$ is the random deviation of the index of refraction which is a function of position r, $k_o^2 = \omega^2/c^2$, c is the sound velocity for a homogeneous medium if μ were everywhere zero, ω is the angular frequency of the source, $u(r)$ is the complex amplitude (for example, the displacement potential), and δ is the Dirac delta function. The time dependence exp $(-i\omega t)$ has heen suppressed. We assume the source is suspended far from any houndaries; that is, we consider the medium to he unhounded.

Our interest is in both the mean field $\langle u \rangle = u_c$ (coherent field) and the fluctuation field $u - u_c = u_i$ (incoherent field), where $\langle \ \rangle$ denotes expectation.

We develop a pair of equations for u_c and u_i as follows. Consider

$$[L + \epsilon L_1 + \epsilon^2 L_2]u = f \tag{2}$$

where L is a linear deterministic operator, L_1 and L_2 are linear stochastic operators, ϵ is a size parameter, and f is a deterministic forcing function. With μ in (1) replaced by $\epsilon\mu$, the correspondence of (1) and (2) is evident. We put $u = u_c + u_i$ into (2) and operate with $\langle \ \rangle$ to ohtain

$$[L + \epsilon\langle L_1 \rangle + \epsilon^2 \langle L_2 \rangle]u_c = f - \epsilon\langle L_1 u_i \rangle - \epsilon^2 \langle L_2 u_i \rangle. \tag{3}$$

We then subtract (3) from (2) in which $u = u_c + u_i$ to obtain

$$[Lu_i + \epsilon(L_1 u_i - \langle L_1 u_i \rangle) + \epsilon^2(L_2 u_i - \langle L_2 u_i \rangle)] = -\epsilon(L_1 - \langle L_1 \rangle)u_c - \epsilon^2(L_2 - \langle L_2 \rangle)u_c . \tag{4}$$

Equation (3) shows the source f of the mean field is countered by the sink $\epsilon\langle L_1 u_i\rangle + \epsilon^2\langle L_2 u_i\rangle$ describing the effects of scattering into the fluctuation field. Equation (4) is not written to exhibit true sources of u_i as much as to exhibit a zero-mean forcing function and zero-mean terms on the left side. These equations are generalizations of those derived by Keller [Ref. 2, p. 166, equations (12) through (13)] for other purposes.

Solution of (3) and (4) can proceed with perturbation theory for the case of small ϵ. Relative to $\epsilon \to 0$, equation (3) exhibits $u_c = O(1)$ and (4) exhibits $u_i = O(\epsilon)$. Accordingly, (3)–(4) can be rewritten

$$[L + \epsilon\langle L_1 \rangle + \epsilon^2\langle L_2 \rangle]u_c = f - \epsilon\langle L_1 u_i \rangle + O(\epsilon^3) \tag{5}$$

$$Lu_i = -\epsilon(L_1 - \langle L_1 \rangle)u_c + O(\epsilon^2). \tag{6}$$

These equations can be partially uncoupled by operating with L^{-1} on (6) and substituting into (5) to obtain

$$[L + \epsilon\langle L_1 \rangle + \epsilon^2\langle L_2 \rangle]u_c = f + \epsilon^2\langle L_1 L^{-1} L_1 \rangle u_c - \epsilon^2\langle L_1 \rangle L^{-1}\langle L_1 \rangle u_c + O(\epsilon^3) \tag{7}$$

$$u_i = -\epsilon L^{-1}(L_1 - \langle L_1 \rangle)u_c + O(\epsilon^2). \tag{8}$$

Equation (7) for the mean field u_c is the result obtained by Keller (Ref. 2, p. 148, equation (10)), who used a successive-substitution solution of (2) in conjunction with a crucial, and at-first-glance mysterious, replacement of $L^{-1}f$ by $\langle u \rangle$. Equation (1.8) is a version of Keller's equation (31) on p. 169 of Ref. 2. Thus, we have shown that these equations arise quite naturally from the pair (3) and (4).

We now specialize (7) and (8) to the case of the Helmholtz equation (1). Here,

$$L = \nabla^2 + k_o^2 \qquad L_1 = 2\mu(r)k_o^2 \qquad L_2 = \mu^2(r)k_o^2 . \tag{9}$$

We assume $\langle \mu(r) \rangle = 0$; that is to say, we neglect any systematic dependence of refractive index upon position (the average profile). We have

$$L^{-1}g = -\int \frac{\exp(ik_o \mid r - r' \mid)}{4\pi \mid r - r' \mid} g(r')\, dr', \tag{10}$$

where the integral is over all space. The inverse L^{-1} is an integral operator with kernel corresponding to the Green's function

$$G(r, r') = -\frac{\exp[ik_o|r - r'|]}{4\pi|r - r'|}. \tag{11}$$

Thus, the pair (7) and (8) is specialized to

$$[\nabla^2 + k_o^2(1 + \epsilon^2\langle\mu^2(r)\rangle)]u_c(r)$$
$$= -\delta(r) - 4\epsilon^2 k_o^4 \int \frac{\exp[ik_o|r - r'|]}{4\pi|r - r'|}\langle\mu(r)\mu(r')\rangle u_c(r')\,dr' + O(\epsilon^3) \tag{12}$$

$$u_i(r) = 2\epsilon k_o^2 \int \frac{\exp[ik_o|r - r'|]}{4\pi|r - r'|}\mu(r')u_c(r')\,dr' + O(\epsilon^2). \tag{13}$$

III. THE AVERAGE FIELD

We now develop an approximation of the solution of (12) for the average field u_c. It is assumed that the refractive index is statistically homogeneous and isotropic. The index covariance function is

$$\Gamma(|r - r'|) = \langle\mu(r)\mu(r')\rangle. \tag{14}$$

Equation (12) becomes

$$\{\nabla^2 + k_o^2[1 + \epsilon^2\Gamma(0)]\}u_c(r)$$
$$= -\delta(r) - 4\epsilon^2 k_o^4 \int \frac{\exp[ik_o|\rho|]}{4\pi|\rho|}\Gamma(|\rho|)u_c(r + \rho)\,d\rho + O(\epsilon^3). \tag{15}$$

We assume an approximation of $u_c(r)$ of the form

$$\frac{\exp[ik|r|]}{4\pi|r|} \tag{16}$$

where k is a constant wave number to be determined ($k \neq k_o$). It will be found that (16) is not a global solution, because a constant k cannot exist. Nevertheless, (16) can serve as a useful local approximation of the solution, with k interpreted as a weak and slowly varying function of $|r|$.

If (16) were the solution, then u_c would satisfy

$$[\nabla^2 + k^2]u_c(r) = -\delta(r). \tag{17}$$

Then (15) and (17) yield

$$\{k_o^2[1 + \epsilon^2\Gamma(0)] - k^2\}u_c(r)$$
$$= -4\epsilon^2 k_o^4 \int \frac{\exp\,[ik_o\,|\,\rho\,|]}{4\pi\,|\,\rho\,|}\,\Gamma(|\,\rho\,|)u_c(r + \rho)\,d\rho + O(\epsilon^3). \qquad (18)$$

The volume integral in (18) can be evaluated by an integration over the surface of a sphere witb radius R followed by a radial integration from $R = 0$ to $R = \infty$. For the surface integration, we need only observe

$$\int_{S:|\rho|=R} u_c(r + \rho)\,\frac{dS}{4\pi R^2}$$
$$= \begin{cases} u_c(r)\,\dfrac{\sin kR}{kR}\,, & 0 < R < |\,r\,| \\ u_c(r)\sin k\,|\,r\,|\,\dfrac{\exp\,[ik(R - |\,r\,|)]}{kR}\,, & R > |\,r\,| \end{cases} \qquad (19)$$

where dS is a differential of area on the sphere $S = \{\rho\colon |\,\rho\,| = R\}$. This mean value theorem follows from (16) and (17); see Appendix A. Then (19) inserted into (18) yields

$$\{k^2 - k_o^2[1 + \epsilon^2\Gamma(0)]\}u_c(r)$$
$$= \frac{4\epsilon^2 k_o^4}{k}\,u_c(r)\Bigg[\int_0^{|r|} \exp\,(ik_oR)\Gamma(R)\sin kR\,dR$$
$$+ \int_{|r|}^{\infty} \exp\,(ik_oR)\Gamma(R)\sin k\,|\,r\,|\exp\,[ik(R - |\,r\,|)]\,dR\Bigg] + O(\epsilon^3). \qquad (20)$$

If (16) were an exact global solution, then $u_c(r)$ could be cancelled in (20); the result would be a relation for the supposedly constant wave number k. But the integrals in (20) suggest that the relation is $|\,r\,|$-dependent, which is a contradiction. Nevertbeless, (16) will serve as a local approximation of $u_c(r)$ in regions in which k is virtually constant.

The following manipulations are made upon tbe integrals in (20). We run the first integral from 0 to ∞ and correct for its contribution from $|\,r\,|$ to ∞ by anotber term in tbe second integral. We tben change tbe variable of integration of the resultant second integral. Then (20) becomes

$$k^2 = k_0^2(1 + \epsilon^2\Gamma(0)) + \frac{4\epsilon^2 k_o^4}{k}\Bigg[\int_0^{\infty} \exp\,(ik_oR)\Gamma(R)\sin kR\,dR$$

$$- \exp [i(k_o - k) \mid r \mid] \int_0^\infty \exp (ik_o R)\Gamma(\mid r \mid + R) \sin kR \, dR \Big] + O(\epsilon^3) \tag{21}$$

The $|r|$-dependence is now confined to the second integral. The large-$|r|$ case occurs when we can assume this integral to be negligible, namely

$$|\exp [i(k_o - k) \mid r \mid]| \, \Gamma(\mid r \mid + R) \ll \Gamma(R), \qquad R \in [0, R_o], \tag{22}$$

where we assume the first integration can run from 0 to R_o with little error. The condition (22) shows that $|r|$ must be much larger than a correlation distance; moreover, (22) shows that the increasing function $\exp[(\mathrm{Im}\, k) \; |r|]$ must be taken into account.

Thus, for large $|r|$, the wave number k satisfies

$$k^2 \approx k_o^2[1 + \epsilon^2\Gamma(0)] + \frac{4\epsilon^2 k_o^4}{k} \int_0^\infty \exp (ik_o R)\Gamma(R) \sin kR \, dR + O(\epsilon^3). \tag{23}$$

This is the relation found by Keller [Ref. 2, p. 151, equation (14)] for the plane-wave problem. As expected, the spherical wave solution far from the source has the same wave number as the plane-wave solution.

The small-$|r|$ case occurs when the integrals in (21) nearly cancel one another; the wave number k is given by

$$k^2 \approx k_0^2[1 + \epsilon^2\Gamma(0)] + O(\epsilon^3) \tag{24a}$$

or

$$k \approx k_o[1 + \tfrac{1}{2}\epsilon^2\Gamma(0)] + O(\epsilon^3). \tag{24b}$$

Whereas (24) yields the small-$|r|$ values of k directly, notice that (23) determines the large-$|r|$ values of k in an implicit fashion. However, an explicit approximation of the large-$|r|$ value of k can be obtained. Notice that (23) could be solved by successive substitutions, the first step employing either k_o in the integral below or employing (24) as follows

$$\frac{4\epsilon^2 k_o^4}{k} \int_0^\infty \exp (ik_o R)\Gamma(R) \sin kR \, dR$$

$$\approx \frac{4\epsilon^2 k_o^4}{k_o[1 + \frac{1}{2}\epsilon^2\Gamma(0)]} \int_0^\infty \exp (ik_o R)\Gamma(R) \sin \{k_o[1 + \tfrac{1}{2}\epsilon^2\Gamma(0)]R\} \, dR$$

$$\approx 4\epsilon^2 k_0^3 \int_0^\infty \exp(ik_oR)\Gamma(R)\{\sin(k_oR)\cos k_o[\tfrac{1}{2}\epsilon^2\Gamma(0)R]$$
$$+ \cos(k_oR)\sin k_o[\tfrac{1}{2}\epsilon^2\Gamma(0)R]\}\, dR$$
$$\approx 4\epsilon^2 k_0^3 \int_0^\infty \exp(ik_oR)\Gamma(R)\sin k_oR\, dR. \tag{25}$$

Since terms have been discarded consistently insofar as powers of ϵ are concerned, approximations (23) and (25) yield

$$k^2 \approx k_o^2[1 + \epsilon^2\Gamma(0)] + 4\epsilon^2 k_o^3 \int_0^\infty \exp(ik_oR)\Gamma(R)\sin k_oR\, dR + O(\epsilon^3). \tag{26}$$

From (26), it follows that

$$k \approx k_o[1 + \tfrac{1}{2}\epsilon^2\Gamma(0)] + 2\epsilon^2 k_o^2 \int_0^\infty \exp(ik_oR)\Gamma(R)\sin k_oR\, dR + O(\epsilon^3), \tag{27}$$

or equivalently

$$\mathrm{Re}\, k \approx k_o[1 + \tfrac{1}{2}\epsilon^2\Gamma(0)] + \epsilon^2 k_o^2 \int_0^\infty \Gamma(R)\sin 2k_oR\, dR + O(\epsilon^3), \tag{28}$$

and

$$\mathrm{Im}\, k \approx \epsilon^2 k_o^2 \int_0^\infty (1 - \cos 2k_oR)\Gamma(R)\, dR + O(\epsilon^3). \tag{29}$$

If Γ has a correlation length L_o and if $k_oL_o \gg 1$ (a large-scale condition not yet imposed), then

$$\mathrm{Im}\, k \approx \epsilon^2 k_o^2 \int_0^\infty \Gamma(R)\, dR. \tag{30}$$

Also, accuracy of the approximation (25) requires the bracketed factor in the integrand to be equivalent to $\sin k_oR$; this holds when

$$k_o\epsilon^2\Gamma(0)L_o \ll 1. \tag{31}$$

But

$$\int_0^\infty \Gamma(R)\, dR \sim \Gamma(0)L_o\,,$$

and (31) is equivalent to

$$\operatorname{Im} k \ll k_o \,. \tag{32}$$

When (32) is not met, neither (28) nor (30) can be expected to be a good approximation. Also, for the successive-substitution procedure to yield a good approximation at this first step, it appears sufficient that the first step value (28) be well approximated by the initial value (24); equivalently,

$$\Gamma(0) \gg k_o \int_0^\infty \Gamma(R) \sin 2k_o R \, dR, \tag{33}$$

which is a restriction on the large-wave number value of an integral which resembles the spectrum of Γ.

The approximation (16) for the average field $u_c(r)$ together with (23) and (24) for the large-$|r|$ and small-$|r|$ values of k comprise the principal results of this section. The further approximations (28) through (30) for the large-$|r|$ case are more useful than (23), but conditions (31) through (33) must be met. When (28) through (30) are compared with the small-$|r|$ approximation (24), it can be seen that the spherical wave (16) develops attenuation and a change in phase velocity as $|r|$ increases. The transition from small-$|r|$ to large-$|r|$ behavior occurs when (22) begins to hold, namely, when the second integral in (21) begins to become negligible. The order of magnitude of this transitional value of $|r|$ is a few correlation lengths.

IV. COVARIANCE OF THE FLUCTUATION FIELD

The previous section provides a solution of (7) or (25) for the average field $u_c(r)$ which now can be used in (8) or (13) to yield the fluctuation field $u_i(r)$. Thus, (11) and (13) yield

$$u_i(r) = -2\epsilon k_o^2 \int G(r, r')\mu(r')u_c(r') \, dr' + O(\epsilon^2), \tag{34}$$

where $u_c(r')$ is given by (16) in which k is a weak function of $|r'|$.

The spatial covariance function $\langle u_i(r)u_i^*(\rho)\rangle$ is now computed for the case in which the medium is statistically homogeneous and isotropic. Equations (34) and (14) yield

$$\langle u_i(r)u_i^*(\rho)\rangle = 4\epsilon^2 k_o^2 \iint G(r, r')G^*(\rho, \rho') \cdot \Gamma(|r' - \rho'|)u_c(r')u_c^*(\rho') \, dr' \, d\rho' + O(\epsilon^3). \tag{35}$$

It is convenient to change to the following variables of integration

(with unity Jacobian):

$$y = \frac{r' + \rho'}{2}, \qquad x = r' - \rho', \tag{36}$$

where

$$r' = y + \frac{x}{2}, \qquad \rho' = y - \frac{x}{2}. \tag{37}$$

Moreover, it is convenient to evaluate the fields at the following points:

$$r = \eta + \frac{\xi}{2}, \qquad \rho = \eta - \frac{\xi}{2}, \tag{38}$$

where, by definition,

$$\eta = \frac{r + \rho}{2}, \qquad \xi = r - \rho. \tag{39}$$

The relation of the positions (36)–(39) is shown in Fig. 1. The covariance of the fluctuation field u_i is thus

$$\left\langle u_i\left(\eta + \frac{\xi}{2}\right)u_i^*\left(\eta - \frac{\xi}{2}\right)\right\rangle$$

$$= 4\epsilon^2 k_o^4 \iint G\left(\eta + \frac{\xi}{2}, y + \frac{x}{2}\right)G^*\left(\eta - \frac{\xi}{2}, y - \frac{x}{2}\right)\Gamma(|x|)$$

$$\cdot u_c\left(y + \frac{x}{2}\right)u_c^*\left(y - \frac{x}{2}\right) dx\, dy + O(\epsilon^3). \tag{40}$$

In words, the second-moment of the fields at observation center η with observation position-difference vector ξ comprises the integrated effect of scattering of the average field by the refractive index at scattering center y with scattering position-difference vector x.

We now approximate the integrand of (40). Although the approxi-

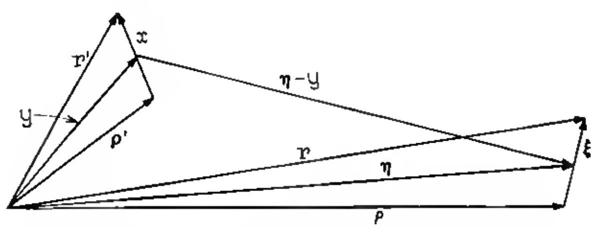

Fig. 1 — Scattering points r', ρ' and receiver points r, ρ.

mations are not valid over all space, they are valid for a region which can account for the major contribution to (40) in the case to be described later. The first approximation involves

$$G(r, r')G^*(\rho, \rho') = \frac{\exp [ik_o(|r - r'| - |\rho - \rho'|)]}{(4\pi)^2 |r - r'| \cdot |\rho - \rho'|}. \tag{41}$$

But

$$\begin{aligned} |r - r'| &= |\eta - y + \tfrac{1}{2}(\xi - x)| \\ &= |\eta - y| + \frac{\frac{1}{2}(\xi - x)\cdot(\eta - y)}{|\eta - y|} + \cdots \end{aligned} \tag{42a}$$

and

$$\begin{aligned} |\rho - \rho'| &= |\eta - y - \tfrac{1}{2}(\xi - x)| \\ &= |\eta - y| - \frac{\frac{1}{2}(\xi - x)\cdot(\eta - y)}{|\eta - y|} + \cdots . \end{aligned} \tag{42b}$$

The above expansions in powers of $(\xi - x)$ are appropriate for a large vector $\eta - y$ as perturbed by the small vectors $\pm\frac{1}{2}(\xi - x)$. Our approximation of (41) is

$$G\left(\eta + \frac{\xi}{2}, y + \frac{x}{2}\right)G^*\left(\eta - \frac{\xi}{2}, y - \frac{x}{2}\right)$$

$$\approx \frac{\exp ik_o\left[\dfrac{(\eta - y)\cdot(\xi - x)}{|\eta - y|}\right]}{(4\pi)^2 |\eta - y|^2} \tag{43}$$

$$= \frac{\exp [ik_s(y)\cdot(\xi - x)]}{(4\pi)^2 |\eta - y|^2} \tag{44}$$

where the relation

$$k_s(y) = k_o \frac{\eta - y}{|\eta - y|} \tag{45}$$

defines a scattering wavevector.

The second approximation involves replacing the coherent-field factor $u_c(y + x/2)u_c^*(y - x/2)$ by a function that locally represents the fields as plane waves. Thus, it follows from (16) that

$$u_c\left(y + \frac{x}{2}\right)u_c^*\left(y - \frac{x}{2}\right) = \frac{\exp\left[i\left(k\left|y + \dfrac{x}{2}\right| - k^*\left|y - \dfrac{x}{2}\right|\right)\right]}{(4\pi)^2\left|y + \dfrac{x}{2}\right|\left|y - \dfrac{x}{2}\right|} \tag{46}$$

where the wavenumber k is a weak function of position. But

$$\left| y + \frac{x}{2} \right| = | y | + \frac{1}{2} \frac{y}{| y |} \cdot x + \cdots \tag{47a}$$

$$\left| y - \frac{x}{2} \right| = | y | - \frac{1}{2} \frac{y}{| y |} \cdot x + \cdots , \tag{47b}$$

leads to the approximation

$$u_c\left(y + \frac{x}{2}\right) u_c^*\left(y - \frac{x}{2}\right) \approx | u_c(y) |^2 e^{ik(y) \cdot x}, \tag{48}$$

where

$$k(y) = (\mathrm{Re}\, k) \frac{y}{| y |} \tag{49}$$

defines an incident wavevector, and

$$| u_c(y) |^2 = \frac{\exp(-2 \,\mathrm{Im}\, k \, | y |)}{(4\pi)^2 \, | y |^2}. \tag{50}$$

Collecting these approximations into (40) yields

$$\left\langle u_i\left(\eta + \frac{\xi}{2}\right) u_i^*\left(\eta - \frac{\xi}{2}\right) \right\rangle$$
$$\approx 4\epsilon^2 k_0^4 \int dy \frac{\exp[ik_s(y) \cdot \xi] \exp[-2 \,\mathrm{Im}\, k \, | y |]}{(4\pi)^4 \, | \eta - y |^2 \, | y |^2}$$
$$\cdot \int dx \, \Gamma(| x |) \exp\{i[k(y) - k_s(y)] \cdot x\}. \tag{51}$$

This is the central result of this section. Equation (51) has the physical interpretation of a volume distribution of sources. The source at y generates a plane wave at the receiver with correlation $\exp[ik_s(y) \cdot \xi]$. The strength of this wave is proportional to $| \eta - y |^{-2}$ $| y |^{-2}$ and to the value of the spectrum

$$S(| \kappa |) = \int dx \, \Gamma(| x |) \exp\{i\kappa \cdot x\} \tag{52}$$

as evaluated at the local wavevector $k(y) - k_s(y)$. At this wave vector, the spectrum is a measure of the amplitude of those components of refractive index with the orientation and the periodicity required for constructive interference (Bragg scattering; compare with Ref. 4, pp. 68–69).

The physical justification of the above approximations follows from (40) by noticing the role played by the index covariance Γ in the integrand. The weighting introduced by Γ means that scattering from center y depends upon the neighborhood of y with linear extent L_o, where L_o is the outer scale. First, the local plane-wave approximation (48) is poorest near the origin where the wavefronts are most curved. Witb a criterion of not more than $\pi/16$ radian departure from plane-wave pbase, Fig. 2 shows tbat $|y|$ must be larger tban $4L_0^2/\lambda$. In fact tbis usual far-field condition can be replaced by $|y| > 4L_o(L_o/\lambda)^{\frac{1}{2}}$ which is less restrictive wben $L_o > \lambda$. This weaker condition, derived in Appendix B, follows from an overbound of tbe phase error in (48) caused by eliminating the remainder of (47). Second, the scattering approximation (43) is poorest near the observation center η where the phase (and amplitude) of (41) can experience large excursions as r', ρ' range over a neighborhood of linear size L_o. A usual far-field condition is $|\eta - y| > 4L_o^2/\lambda$ or $|\eta - y| > 4(|\xi| + L_o)^2/\lambda$. Again, wben $L_o > \lambda$, only a weaker condition,

$$|\eta - y| > 4(|\xi| + L_o)\left(\frac{|\xi| + L_o}{\lambda}\right)^{\frac{1}{2}}, \tag{53}$$

need be met. Condition (53) follows from an overbound of the pbase error in (43) associated with the remainder in (52), (see Appendix B). Strictly speaking, the y-integration in (51) must exclude the near-source and near-receiver spheres of radius $4L_o(L_o/\lambda)^{\frac{1}{2}}$, and their contributions must be evaluated separately. In Section V we give a condition necessary for this contribution to be negligible.

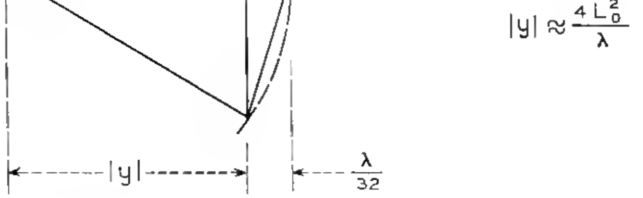

Fig. 2 — Distance for the plane-wave approximation [in fact, only $4L_o(L_o/\lambda)^{1/2}$ required].

Apart from the excluded regions of integration, the validity of approximation (51) does not rely upon a "large-scale" condition requiring the wavelength λ to be much smaller than some refractive-index scale size. But when such a condition is met, (51) yields both a maximum angle of important scattering and a finite volume of important scattering. In the approximation (51), the refractive-index spectrum (52) is evaluated at the local wave vector,

$$k(y) - k_s(y). \tag{54}$$

Suppose there exists an inner scale l_o such that for $|\kappa| > 2\pi/l_o$ the spectrum (52) is negligible. Since the maximum magnitude (54) can attain is of order $4\pi/\lambda$, whereas $2\pi/\lambda \gg 2\pi/l_o$, it follows that the integrand of (51) is large only for values of y such that

$$|k(y) - k_s(y)| < 2\pi/l_o\,. \tag{55}$$

Under the assumption that $|k(y)| = k_o = 2\pi/\lambda$, condition (55) yields the maximum angle of important scattering. With $\psi(y)$ the angle between $k(y)$ and $k_s(y)$, as shown in Fig. 3, we have

$$|k(y) - k_s(y)|^2 = 2k_o^2 - 2k_o^2 \cos \psi(y) = 4k_o^2 \sin^2 \frac{\psi(y)}{2}. \tag{56}$$

Then (55) and (56) yield $\cos \psi(y) > 1 - \lambda^2/2l_o^2$ or $2l_o/\lambda \sin \psi(y)/2 < 1$ or approximately $\psi(y) < \lambda/l_o$.

These conditions may be used to find the region of important scattering. Figure 4 shows cylindrical coordinates with origin at the midpoint between transmitter and receiver; there is rotational symmetry around the transmitter-receiver axis. With $\tan \psi$ constant, we have

$$\tan \psi = \tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}. \tag{57}$$

But

$$\tan \alpha = \frac{b}{\frac{L}{2} + a} \tag{58}$$

$$\tan \beta = \frac{b}{\frac{L}{2} - a}. \tag{59}$$

Algebraic manipulations which include completing a square yield

$$a^2 + \left(b + \frac{L}{2 \tan \psi}\right)^2 = \left(\frac{L}{2 \sin \psi}\right)^2. \tag{60}$$

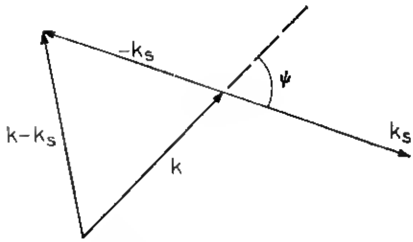

Fig. 3 — Angle of scattering.

Equation (60) is a circle in the a-b plane passing through the transmitter and receiver-center locations, Fig. 5. The slope of the curve is

$$\frac{db}{da} = -\frac{a}{b + \dfrac{L}{2\tan\psi}}. \tag{61}$$

Thus, near the transmitter,

$$\frac{b}{\dfrac{L}{2} + a} \approx \left.\frac{db}{da}\right|_{(-L/2,0)} = \tan\psi, \tag{62}$$

and near the receiver

$$\frac{b}{\dfrac{L}{2} - a} \approx -\left.\frac{db}{da}\right|_{(L/2,0)} = \tan\psi. \tag{63}$$

Also, at the midpoint $a = 0$, (60) yields

$$b = \frac{L}{2}\left(\frac{1}{\sin\psi} - \frac{1}{\tan\psi}\right) = \frac{L}{2}\tan\frac{\psi}{2}. \tag{64}$$

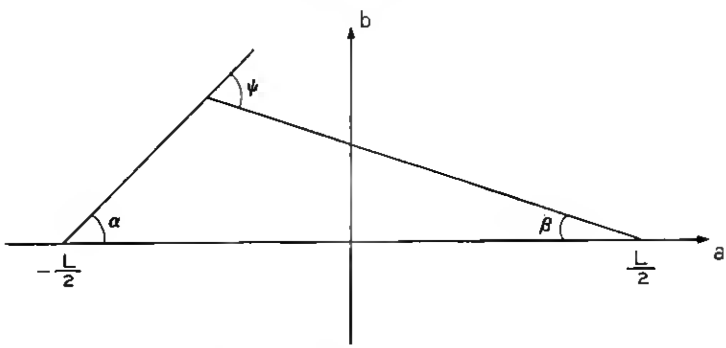

Fig. 4 — Coordinates for the region of important scattering.

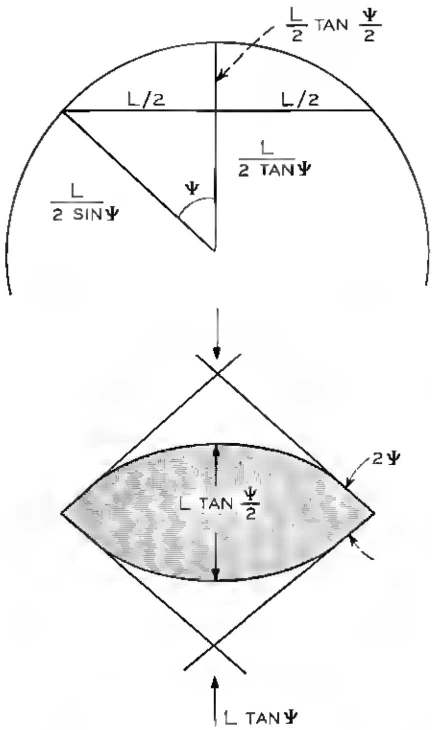

Fig. 5 — Region of important scattering (spherical waves).

The condition for important scattering is

$$\cos \psi > 1 - \frac{\lambda^2}{2l_o^2} \triangleq \cos \Psi. \tag{65}$$

The volume specified by (65) is enclosed within the surface generated by rotating the arc of the circle (60), with $\psi = \Psi$, around the transmitter-receiver axis. This volume lies within the volume common to two cones with apexes at transmitter and receiver, each cone with half-angle Ψ.

For the large-scale case, $\lambda \ll l_o \leqq L_o$, a condition necessary for (51) to be an accurate approximation of the covariance (40) is now apparent. The volume of important scattering shown in Fig. 5 must be much larger than the volumes in which the integrand of (51) is a poor approximation of the integrand of (40). These comprise a near-source cone of axial length $4L_o^{3/2}/\lambda^{\frac{1}{2}}$ and a near-receiver cone of axial length $4(L_o + |\xi|)^{3/2}/\lambda^{\frac{1}{2}}$.

An equivalent condition is seen to be that the transmitter-receiver distance must be much larger than the axial lengths of these cones, that is to say,

$$L = | \eta | \gg 4(L_o + | \xi |)^{3/2}/\lambda^{\frac{1}{2}}. \tag{66}$$

It remains to observe that the covariance expression (40) is itself an accurate relation provided that the average field is not severely attenuated by virtue of scattering into the fluctuation field. The attenuation exhibited in (50), as evaluated throughout the above region of important scattering, must be small; that is to say,

$$(\mathrm{Im}\, k) \mid \eta \mid \ll 1, \tag{67}$$

where Im k is given by (30). Combining conditions (66) and (67) yields an interval for validity of (51). When $| \xi | = 0$, this interval is

$$L_o^{3/2}/\lambda^{\frac{1}{2}} \ll | \eta | \ll (\mathrm{Im}\, k)^{-1}. \tag{68}$$

In other words, the transmitter-to-receiver distance must be (*i*) sufficiently large so that far-field approximations of the covariance are valid, and (*ii*) sufficiently small so that single-scatter perturbation approximations are valid.

V. REDUCTION OF THE INTEGRATION FOR THE COVARIANCE-SPHERICAL AND PLANAR CASES

5.1 *Spherical-Wave Case*

The central result of the previous section is the approximation (51) of the covariance. The problem remains to evaluate the integral specified by this approximation. In this section, we introduce a set of coordinates which simplifies the integration, the result being (77). Although Section V indicates the extent of the important region of integration, the result (77) is equivalent to integration over all space rather than over only the important region. Under the large-scale approximation, the formula (77) is specialized to (83) and to (85) for transverse and longitudinal receiver separations.

For simplicity, we first observe that (51) can be replaced by an expression employing the unperturbed field u_o rather than the average field u_c. We need only observe from (7) that

$$u_c = L^{-1}f + O(\epsilon^2)$$

where $\langle L_1 \rangle = 0$. It immediately follows that (8) can be replaced by

$$u_i = -\epsilon L^{-1} L_1 u_o + O(\epsilon^2)$$

where $u_o = L^{-1}f$ is the field that would exist in the nonrandom medium ($\epsilon = 0$). In our special case,

$$u_o(r) = \frac{\exp [ik_o \mid r \mid]}{4\pi \mid r \mid},$$

and accordingly

$$u_i(r) = -2\epsilon k_o^2 \int G(r, r')\mu(r')u_o(r')\, dr' + O(\epsilon^2)$$

can replace (34). Equivalently, (51) can be approximated by

$$\left\langle u_i\left(n + \frac{\xi}{2}\right)u_i^*\left(\eta - \frac{\xi}{2}\right)\right\rangle$$

$$\approx 4\epsilon^2 k_o^4 \int dy \frac{\exp [ik_s(y)\cdot\xi]}{(4\pi)^4 \mid y \mid^2 \mid \eta - y \mid^2} S(\mid k_o\hat{y} - k_s(y) \mid), \qquad (69)$$

where the spectrum S is defined by (52) and where $\hat{y} = y/\mid y \mid$. That is to say, the replacement of u_s by u_o corresponds to the replacement of $k(y)$ by $k_o\hat{y}$.

The volume integration can be carried out with spherical coordinates which have the receiving center η as their origin. In such coordinates, the differential of volume of $d\Omega dR\, R^2$, where $R = \mid \eta - y \mid$ and $d\Omega$ is the differential of the solid angle. Since $k_s(y)$ is a function only of the direction of an element $d\Omega$ relative to the origin at η, it follows that (69) equals

$$\frac{4\epsilon^2 k_o^4}{(4\pi)^4} \int d\Omega \exp [ik_s\cdot\xi] \int dR \,\{\mid y \mid^{-2} S(\mid k_o\hat{y} - k_s(y) \mid)\} \qquad (70)$$

where the factor in braces is to be evaluated as a function of R with k_s fixed.

The angular integration in (70) will use the coordinates in Fig. 6, where $\theta = 0$ corresponds to the direction of the transmitter. The radial integration in (70) will employ the angle ψ shown in Fig. 7. The argument of the spectrum is the square root of

$$\mid k_o\hat{y} - k_s \mid^2 = k_o^2(2 - 2 \cos \psi) = 4k_o^2 \sin^2 \frac{\psi}{2}. \qquad (71)$$

The law of sines is

$$\frac{\mid y \mid}{\sin \theta} = \frac{L}{\sin (\pi - \psi)} = \frac{R}{\sin (\psi - \theta)}, \qquad (72)$$

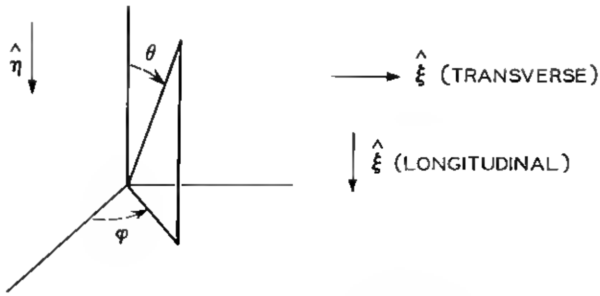

Fig. 6 — Spherical polar coordinates at the receiver.

and thus

$$|y| = L\frac{\sin\theta}{\sin\psi} \tag{73}$$

$$\frac{dR}{d\psi} = L\frac{\sin\theta}{\sin^2\psi}. \tag{74}$$

The radial integral in (70) is thus

$$(L\sin\theta)^{-1}\int_\theta^\pi d\psi\; S\left(2k_o\sin\frac{\psi}{2}\right), \tag{75}$$

and (70) becomes

$$\frac{4\epsilon^2k_o^4L}{(4\pi)^2(4\pi L)^2}\int_0^\pi d\theta\int_0^{2\pi}d\varphi\,\exp\,(ik_s\cdot\xi)\int_\theta^\pi d\psi\; S\left(2k_o\sin\frac{\psi}{2}\right). \tag{76}$$

The θ-ψ integration is over the triangle $\{0 \leqq \theta \leqq \pi, \theta \leqq \psi \leqq \pi\} = \{0 \leqq \psi \leqq \pi, 0 \leqq \theta \leqq \psi\}$, so that interchange of the order of integration yields

$$\frac{\epsilon^2k_o^4L}{(2\pi)^2(4\pi L)^2}\int_0^\pi d\psi\; S\left(2k_o\sin\frac{\psi}{2}\right)\int_0^\psi d\theta\int_0^{2\pi}d\varphi\,\exp\,(ik_s\cdot\xi). \tag{77}$$

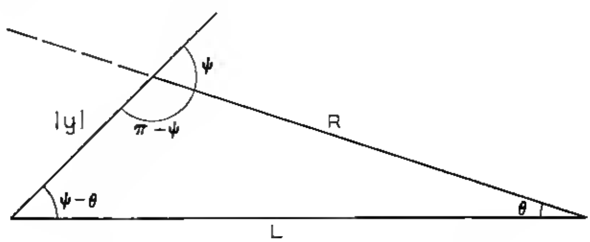

Fig. 7 — Radial variable R related to angle ψ.

Further specialization of (77) is made to the cases in which the receiving displacement ξ is transverse and is longitudinal, Fig. 6. In the transverse case, $k_s \cdot \xi = -k_o \mid \xi \mid \sin \theta \sin \varphi$, and (77) becomes

$$\frac{\epsilon^2 k_o^4 L}{2\pi(4\pi L)^2} \int_0^\pi d\psi \; S\left(2k_o \sin \frac{\psi}{2}\right) \int_0^\psi d\theta \; J_o(k_o \mid \xi \mid \sin \theta). \tag{78}$$

In the longitudinal case, $k_s \cdot \xi = k_o \mid \xi \mid \cos \theta$, and (77) becomes

$$\frac{\epsilon^2 k_o^4 L}{2\pi(4\pi L)^2} \int_0^\pi d\psi \; S\left(2k_o \sin \frac{\psi}{2}\right) \int_0^\psi d\theta \exp\,(ik_o \mid \xi \mid \cos \theta). \tag{79}$$

Expressions (77) to (79) correspond to integration over all space, rather than over only the region of important scattering. Further approximations rely upon the cutoff provided by $S(\kappa)$ for $\kappa > 2\pi/l_o$, where l_o is the inner scale size and $l_o \gg \lambda$. For the transverse case, (78) becomes

$$\frac{\epsilon^2 k_o^2 L}{2\pi(4\pi L)^2} \int_0^\infty dx \; S(x) \int_0^x d\kappa \; J_o(\kappa \mid \xi \mid), \tag{80}$$

and in the longitudinal case, (79) yields

$$\frac{\epsilon^2 k_o^2 L}{2\pi(4\pi L)^2} \exp\,(ik_o \mid \xi \mid) \int_0^\infty dx \; S(x) \int_0^x d\kappa \exp\left(-i \frac{\mid \xi \mid \kappa^2}{2k_o}\right). \tag{81}$$

The κ-integral in (80) can be evaluated in closed form, namely

$$\int_0^x d\kappa \; J_o(\kappa \mid \xi \mid) = xJ_o(x \mid \xi \mid) + \frac{\pi}{2} xJ_1(x \mid \xi \mid)H_o(x \mid \xi \mid) - \frac{\pi}{2} xJ_o(x \mid \xi \mid)H_1(x \mid \xi \mid) \tag{82}$$

where H_ν are the Struve functions. Thus, for the transverse case, (80) is

$$\frac{\epsilon^2 k_o^2 L}{2\pi(4\pi L)^2} \int_0^\infty dx \; xS(x)\left[J_o(x \mid \xi \mid) + \frac{\pi}{2} J_1(x \mid \xi \mid)H_o(x \mid \xi \mid) - \frac{\pi}{2} J_o(x \mid \xi \mid)H_1(x \mid \xi \mid)\right]. \tag{83}$$

The κ-integral in (81) is related to the Fresnel integrals, namely

$$\int_0^x d\kappa \exp\left(-i \frac{\mid \xi \mid \kappa^2}{2k_o}\right) = \left(\frac{\pi k_o}{\mid \xi \mid}\right)^{\frac{1}{2}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{(x^2 \mid \xi \mid /2k_o)^{\frac{1}{2}}} dt \exp\,(-it^2)$$
$$= \left(\frac{\pi k_o}{\mid \xi \mid}\right)^{\frac{1}{2}} \left[\hat{C}\left(\frac{x^2 \mid \xi \mid}{2k_o}\right)^{\frac{1}{2}} - i\hat{S}\left(\frac{x^2 \mid \xi \mid}{2k_o}\right)^{\frac{1}{2}}\right]. \tag{84}$$

Thus, for the longitudinal case, (81) is

$$\frac{\epsilon^2 k_o^2 L}{2\pi(4\pi L)^2} \exp(ik_o \mid \xi \mid) \int_0^\infty dx\, xS(x)\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(\frac{2k_o}{x^2 \mid \xi \mid}\right)^{\frac{1}{2}} \cdot\left[\hat{C}\left(\frac{x^2 \mid \xi \mid}{2k_o}\right)^{\frac{1}{2}} - i\hat{S}\left(\frac{x^2 \mid \xi \mid}{2k_o}\right)^{\frac{1}{2}}\right]. \tag{85}$$

5.2 *Plane-Wave Case*

For the spherical-wave case, the spatial covariance $\langle u_i(\eta + \xi/2)u_i^*(\eta - \xi/2)\rangle$ is given by (77) and its transverse and longitudinal specializations (80) and (81) or (83) and (85). By way of contrast, we derive the corresponding expressions for the plane-wave case.

Approximations (45) and (48) show that the covariance (40) is approximately

$$\left\langle u_i\left(\eta + \frac{\xi}{2}\right)u_i^*\left(\eta - \frac{\xi}{2}\right)\right\rangle \approx 4\epsilon^2 k_o^4 \int dy \frac{\exp[ik_s(y)\cdot\xi]}{(4\pi)^2 \mid \eta - y \mid^2} S(\mid k(y) - k_s(y) \mid) \tag{86}$$

where $k(y)$ is a constant wavevector, $\mid k(y) \mid = k_o$ in keeping with the interchange of u_e and u_o, and $\mid u_o(y) \mid^2 = 1$. Here the integral is over the volume of a half-space with k perpendicular to the face.

With spherical coordinates centered at the receiving center η, (86) becomes

$$\frac{\epsilon^2 k_o^4}{(2\pi)^2} \int d\Omega \exp(ik_s\cdot\xi)S(\mid k - k_s \mid) \int dR \tag{87}$$

where the radial integral has k_s-dependent integration limits corresponding to the half-space interface. Under the large-scale approximation, $\lambda/l_o \ll 1$, the radial integral is approximately L. For transverse separation, (87) is

$$\frac{\epsilon^2 k_o^4 L}{(2\pi)^2} \int_0^\pi \int_0^{2\pi} d\theta\, d\varphi \sin\theta \exp(-ik_o \mid \xi \mid \sin\theta \sin\varphi) S\left(2k_o \sin\frac{\theta}{2}\right) \tag{88}$$

or

$$\frac{\epsilon^2 k_o^4 L}{2\pi} \int_0^\pi d\theta \sin\theta J_o(k_o \mid \xi \mid \sin\theta) S\left(2k_o \sin\frac{\theta}{2}\right). \tag{89}$$

Under the large-scale approximation, with small angles yielding the

significant part of (89) the covariance becomes

$$\frac{\epsilon^2 k_o^2 L}{2\pi} \int_0^\infty dx\, x S(x) J_o(x \mid \xi \mid). \tag{90}$$

This expression was obtained by Tatarski [Ref. 4, equation (7.64)] to be equal to twice the correlation function for either the log-amplitude or the phase fluctuation of the total field. But

$$S(x) = \frac{4\pi}{x} \int_0^\infty dr\, r \Gamma(r) \sin xr, \tag{91}$$

because (91) is a function of $|\kappa|$, and

$$\int_0^\infty dx \sin(xr) J_o(x \mid \xi \mid) = \begin{cases} (r^2 - |\xi|^2)^{-1/2}, & r^2 > |\xi|^2, \\ 0 \qquad , & r^2 < |\xi|^2. \end{cases} \tag{92}$$

Substituting (91) and (92) into (90) and changing the variable of integration shows that

$$\left\langle u_i\left(\eta + \frac{\xi}{2}\right) u_i^*\left(\eta - \frac{\xi}{2}\right) \right\rangle \approx 2\epsilon^2 k_o^2 L \int_0^\infty dr\; \Gamma(r^2 + |\xi|^2)^{\frac{1}{2}} \tag{93}$$

for transverse separation. This is a central result of much of the literature (for example, Ref. 3); we have obtained this result in a simple and novel way.

For the case of longitudinal separation, (87) is

$$\frac{\epsilon^2 k_o^4 L}{(2\pi)^2} \int_0^\pi \int_0^{2\pi} d\theta\, d\varphi \sin\theta \exp(ik_o \mid \xi \mid \cos\theta) S\left(2k_o \sin\frac{\theta}{2}\right) \tag{94}$$

or

$$\frac{\epsilon^2 k_o^4 L}{2\pi} \int_0^\pi d\theta\, 2 \sin\frac{\theta}{2} \cos\frac{\theta}{2} \exp\left[ik_o \mid \xi \mid \left(1 - 2\sin^2\frac{\theta}{2}\right)\right] S\left(2k_o \sin\frac{\theta}{2}\right). \tag{95}$$

Under the large-scale approximation, the upper limit of the variable of integration $\kappa = 2k_o \sin\theta/2$ can be replaced by infinity. Thus,

$$\left\langle u_i\left(\eta + \frac{\xi}{2}\right) u_i^*\left(\eta - \frac{\xi}{2}\right) \right\rangle \approx \frac{\epsilon^2 k_o^2 L}{2\pi} \exp(ik_o \mid \xi \mid) \int_0^\infty dx\, x S(x) \exp\left(-i\frac{\mid \xi \mid x^2}{2k_o}\right). \tag{96}$$

It does not appear possible to simplify (96) by using (91) together

with the sine-transform of the exponential in (96). However, the special case

$$\Gamma(r) = \exp(-r^2/2l_o^2) \tag{97}$$

is of interest. Then

$$S(x) = (2\pi)^{3/2} l_o^3 \exp(-l_o^2 x^2/2). \tag{98}$$

Inserting (98) into (96) and using the variable $u = l_o^2 x^2/2$ for integration yields

$$\left\langle u_i\left(\eta + \frac{\xi}{2}\right) u_i^*\left(\eta - \frac{\xi}{2}\right)\right\rangle \approx \epsilon^2 k_o^2 L l_o (2\pi)^{\frac{1}{2}} \exp(ik_o \mid \xi \mid) \frac{1 - i \dfrac{\mid \xi \mid}{k_o l_o^2}}{1 + \left(\dfrac{\mid \xi \mid}{k_o l_o^2}\right)^2}. \tag{99}$$

This expression corresponds to a result of Chernov [Ref. 1, p. 94, equation (187)] for longitudinal log-amplitude or phase fluctuations. Tbe magnitude of the last factor in (99) is reduced by $5^{\frac{1}{2}}$ when $\mid \xi \mid = 2k_o l_o^2$.

5.3 *Comparison of the Spherical and Planar Cases*

Notice that the relative variance (zero receiver separation) is the same for the spherical and planar cases. That is to say, expressions (80) and (81) yield the same variance, relative to the spherical-wave power $(4\pi L)^{-2}$, as do expressions (90) and (96), relative to the unity plane-wave power.

For transverse receiver separation, the planar-case result (90) can be compared with the spherical-case result (80). The weighting of the spectral function $xS(x)$ is $j_o(x \mid \xi \mid)$ in the planar case; this $\mid \xi \mid$-function has its first zero at $\mid \xi \mid = 2.4/x$ with subsequent zeros spaced $3.1/x$ apart. In the spherical case, the weighting is

$$x^{-1} \int_0^x d\kappa \, J_o(\kappa \mid \xi \mid);$$

this $\mid \xi \mid$-function is a mixture of functions with "periodicity" larger than that of $J_o(x \mid \xi \mid)$. Presumably, correlation lengths would usually be larger in the spherical case than in the planar case.

For longitudinal receiver separation, the planar-case result (96) can be compared with the spherical-case result (81). The weighting of $xS(x)$ is

$$\exp\left(-i \frac{\mid \xi \mid x^2}{2k_o}\right)$$

in the planar case; this $|\xi|$-function has period $4\pi k_o/x^2$. In the spherical case, the weighting is

$$x^{-1}\int_0^x d\kappa \exp\left(-i\frac{|\xi|\kappa^2}{2k_o}\right);$$

this $|\xi|$-function is a mixture of longer-period functions, and again correlation lengths would presumably be larger in the spherical case than in the planar case.

Physical reasoning also suggests that correlation lengths are larger in the spherical case than in the planar case. First, compare the regions of important scattering. For the spherical case, this region is sketched in Fig. 5, where the angle Ψ is given by (65). For the planar case, this region is a cone with half-angle Ψ and axial length L (the transmitter-receiver separation being replaced by the distance the receiver is imbedded into a half-space of random refractive index), Fig. 8. Comparison of the two regions suggests the the fluctuation field in the spherical case is more directive than the fluctuation field in the planar case.

Second, consider the implication of a more directive fluctuation field. The directionality function N can be defined by

$$\frac{\left\langle u_i\left(\eta+\frac{\xi}{2}\right)u_i^*\left(\eta-\frac{\xi}{2}\right)\right\rangle}{\langle|u_i(\eta)|^2\rangle} = \int d\Omega \exp(ik_s\cdot\xi)N(k_s). \tag{100}$$

A wave in direction k_s contributes a correlation $\exp(ik_s\cdot\xi)$, and the total correlation is a weighted average of such constituents. The form (100) is exhibited by (70) in the spherical case and by (87) in the planar case. An idealized directionality function would be constant with $k_s\cdot\eta$ above a threshold and would be zero elsewhere. That is to say, (100) would be

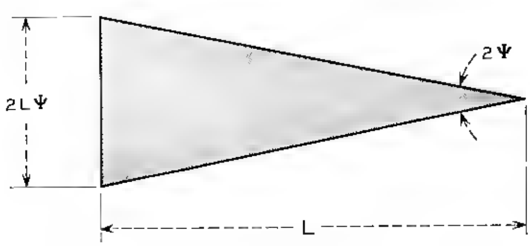

Fig. 8 — Region of important scattering (plane waves).

$$\frac{1}{\Delta\Omega}\int_{\Delta\Omega} d\Omega \exp (ik_s\cdot\xi) \tag{101}$$

where $\Delta\Omega$ is a small cap on the unit sphere of size $2\pi(1 - \cos\Theta) \approx \pi\Theta^2$. When ξ is transverse, (101) becomes

$$[2\pi(1 - \cos\Theta)]^{-1}\int_0^\Theta\int_0^{2\pi} d\theta\, d\varphi \sin\theta \exp(-ik_o\,|\,\xi\,|\sin\theta\sin\varphi) \tag{102}$$

$$(1 - \cos\Theta)^{-1}\int_0^\Theta d\theta \sin\theta J_o(k_o\,|\,\xi\,|\sin\theta).$$

Under the small-angle approximation, (102) yields

$$\frac{2J_1(k_o\,|\,\xi\,|\,\Theta)}{k_o\,|\,\xi\,|\,\Theta}. \tag{103}$$

The correlation function (103) is unity at $|\,\xi\,| = 0$, is 0.88 at $|\,\xi\,| = \lambda/2\pi\Theta \approx 0.16\,\lambda/\Theta$, and is zero at $|\,\xi\,| \approx 0.61\,\lambda/\Theta$.

When ξ is longitudinal, (101) becomes

$$[2\pi(1 - \cos\Theta)]^{-1}\int_0^\Theta\int_0^{2\pi} d\theta\, d\varphi \sin\theta \exp(ik_o\,|\,\xi\,|\cos\theta)$$

$$= \exp(ik_o\,|\,\xi\,|)\,\frac{1 - \exp[ik_o\,|\,\xi\,|(1 - \cos\theta)]}{ik_o\,|\,\xi\,|(1 - \cos\Theta)} \tag{104}$$

$$= \exp(ik_o\,|\,\xi\,|)\,\frac{1 - \exp[-i2k_o\,|\,\xi\,|\,\Delta\Omega/4\pi]}{i2k_o\,|\,\xi\,|\,\Delta\Omega/4\pi}. \tag{105}$$

The correlation $\exp(ik_o\,|\,\xi\,|)$ associated with a plane wave is modulated by a function having ripple in its numerator with period $\lambda(2\pi/\Delta\Omega)$.

The λ-dependence exhibited in (103) and (107) must be tempered by the λ-dependence of the angle Θ. Figures 5 and 8 suggest that Θ would be at most λ/l_o ($\Delta\Omega$ at most $\pi\lambda^2/l_o^2$) for equivalence of the idealized and true directionality functions. For the transverse case, the null of (103) would be at $|\,\xi\,| = 0.61\,l_o$ or more; the λ-dependence disappears as in (80) and (90). For the longitudinal case, the period in (105) would be at least $2l_o^2/\lambda$; this period is to be compared with the width of the last factor in (99) for the plane-wave gaussian-index correlation case.

Both (103) and (105) exhibit the fact that correlation lengths are inversely proportional to $\Delta\Omega$, the width of the directionality function. But the scattering volumes depicted in Figs. 5 and 8 show that this width is smaller in the spherical case than in the planar case. This

physical reasoning corroborates the previous interpretation of the integration formulas which showed larger correlation lengths for the spherical case.

VI. EXAMPLES OF TRANSVERSE COVARIANCES

It has been sbown that, for transverse receiver separation, the covariance $\langle u_i(\eta + \xi/2)u_i^*(\eta - \xi/2)\rangle$ is given by (80) in the spherical case and by (90) in the planar case. These expressions are now evaluated in closed form for two illustrative spectra.

Recall that the spectrum $S(\kappa)$ is related to the refractive index covariance $\Gamma(r)$ by (52) whicb becomes (91) for the statistically isotropic case. For convenience, (91) is repeated here, together with its inverse:

$$S(\kappa) = \frac{4\pi}{\kappa}\int_0^\infty dr\, r\Gamma(r)\sin\kappa r \tag{106}$$

$$\Gamma(r) = \frac{1}{2\pi^2 r}\int_0^\infty d\kappa\, \kappa S(\kappa)\sin\kappa r. \tag{107}$$

Also, the planar-case covariance (90) is

$$\frac{\epsilon^2 k_o^2 L}{2\pi}\int_0^\infty d\kappa\, \kappa S(\kappa) J_o(\kappa \mid \xi \mid), \tag{108}$$

and the spherical-case covariance (80) is

$$\frac{\epsilon^2 k_o^2 L}{(4\pi L)^2 2\pi}\int_0^\infty d\kappa\, J_o(\kappa \mid \xi \mid)\int_\kappa^\infty dx\, S(x); \tag{109}$$

the order of integration has been changed.

The normalization of the spectrum follows from (107) evaluated at $r = 0$,

$$1 = \frac{1}{2\pi^2}\int_0^\infty d\kappa\, \kappa^2 S(\kappa), \tag{110}$$

so that ϵ^2 plays the role of the variance of the refractive index.

Our first example is the case of an exponential spectrum:

$$S(\kappa) = \pi^2\Lambda^3 \exp[-\Lambda\kappa], \tag{111}$$

$$\Gamma(r) = [1 + (r/\Lambda)^2]^{-2} \tag{112}$$

where $\Lambda \gg \lambda$. Then, the planar covariance (108) is

$$\frac{\epsilon^2 k_o^2 L \pi^2 \Lambda}{2\pi}[1 + (\mid \xi \mid/\Lambda)^2]^{-3/2} \tag{113}$$

and the spherical covariance (109) is

$$\frac{\epsilon^2 k_o^2 L\pi^2 \Lambda}{(4\pi L)^2 2\pi}\,[1 + (|\,\xi\,|/\Lambda)^2]^{-1/2}. \tag{114}$$

The respective correlation functions in (112) to (114) are plotted in Fig. 9. The correlation length for the spherical case is larger than the comparable correlation lengths for the planar case and for the refractive index.

Our second example is the case of a constant spectrum:

$$S(\kappa) = \begin{cases} \dfrac{3\Lambda^3}{4\pi}, & 0 \leqq \kappa \leqq 2\pi/\Lambda \\ 0, & \kappa > 2\pi/\Lambda \end{cases} \tag{115}$$

$$\Gamma(r) = \frac{\sin\dfrac{2\pi r}{\Lambda} - \dfrac{2\pi r}{\Lambda}\cos\dfrac{2\pi r}{\Lambda}}{\dfrac{1}{3}\left(\dfrac{2\pi r}{\Lambda}\right)^3}. \tag{116}$$

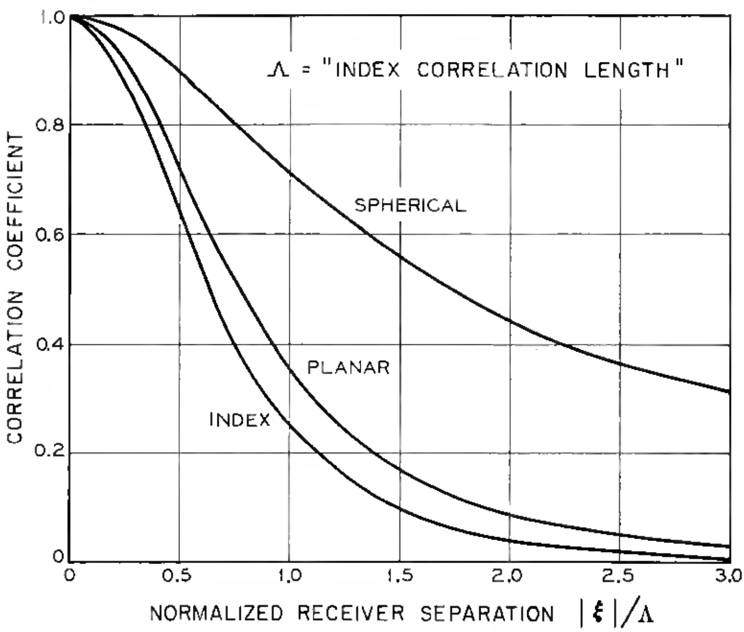

Fig. 9 — Correlations for the exponential spectrum.

Then, the planar covariance (108) becomes

$$\frac{\epsilon^2 k_o^2 L}{2\pi} \frac{3\pi\Lambda}{2} \left[\frac{2J_1\left(\frac{2\pi \mid \xi \mid}{\Lambda}\right)}{2\pi \mid \xi \mid / \Lambda} \right]. \tag{117}$$

The correlation function in brackets agrees with (103) with $\Theta = \lambda/\Lambda$, which determines the angular extent of the constant directionality function. The spherical covariance (109) is

$$\frac{\epsilon^2 k_o^2 L}{(4\pi L)^2} \frac{3\pi\Lambda}{2} \left\{ 2J_o\left(\frac{2\pi \mid \xi \mid}{\Lambda}\right) - \frac{2J_1\left(\frac{2\pi \mid \xi \mid}{\Lambda}\right)}{2\pi \mid \xi \mid / \Lambda} \right.$$
$$\left. + \pi J_1\left(\frac{2\pi \mid \xi \mid}{\Lambda}\right) H_o\left(\frac{2\pi \mid \xi \mid}{\Lambda}\right) - \pi J_o\left(\frac{2\pi \mid \xi \mid}{\Lambda}\right) H_1\left(\frac{2\pi \mid \xi \mid}{\Lambda}\right) \right\}, \tag{118}$$

where H_ν are Struve functions. The correlation functions in (116) to (118) are plotted in Fig. 10. As before, the correlation length for the spherical case is larger than the comparable correlation lengths for the planar case and for the refractive index.

VII. SUMMARY

In Section II, the perturbation theory of Keller is developed in a novel way.[2] This development shows that the nearly uncoupled equations (12) and (13) arise naturally from the fundamental pair (3) and (4). In Section III, the equation for the average field (12) is solved for the case in which the refractive index is statistically homogeneous and isotropic. The spherical wave (16) is shown to be a good local approximation of the average field; the wavenumber k is a weak function of position and satisfies (21). Beyond a few correlation lengths from the source, the wavenumber is a constant given approximately by (28) to (30).

In Section IV, the equation for the fluctuation field (13) is sbown to imply (40) for the spatial covariance of the field. A useful approximation of the covariance is (51) which is then justified on physical grounds. For the large-scale case $\lambda \ll l_o$, where λ is the wavelength and l_o is the inner scale for the refractive index, this approximation shows that the region of important scattering is given by (60) with (65) and lies within tbe volume common to two cones with apexes at transmitter and receiver, each cone with half-angle approximately λ/l_o. The interval for validity of (51) is given by (68) which states that the transmitter-to-receiver distance is sufficiently large for far-field covariance approxima-

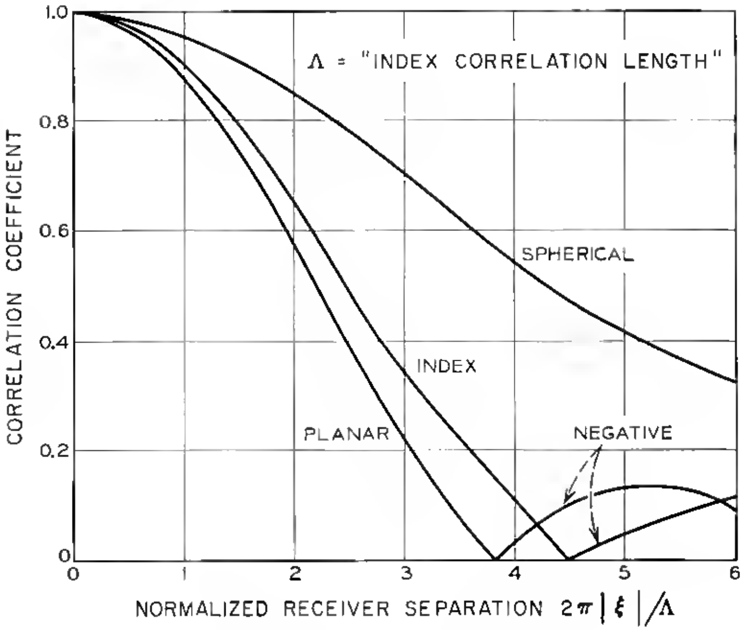

Fig. 10 — Correlations for the constant spectrum.

tions to be valid but is sufficiently small for single scatter perturbation approximations to be valid. In Appendix B, it is shown that a far-field condition relative to the covariance is less restrictive than a far-field condition relative to the field itself.

In Section V, the volume integration of (51) for the covariance is transformed to the angular integrations exhibited in (77). For the large-scale case, the covariance is given by (80) and (81) for transverse and longitudinal separation of receivers. These expressions for our spherical-wave model are contrasted with (90) and (96) for the plane-wave model, showing that relative variances are the same but that correlation lengths are larger in the spherical-wave case than in the plane-wave case. This is to be expected on physical grounds, for comparison of the volumes of important scattering for the two cases indicates that the fluctuation field is more directive in the spherical case. But a more directive field has longer correlation lengths; this is illustrated by (103) and (105) for transverse and longitudinal separations under the idealized directionality function in (101). In Section VI, two special cases of refractive-index correlation which cor-

respond to an exponential spectrum and a constant spectrum are considered. For transverse separation, the covariance functions are derived in closed form. Plots of the correlation functions show that correlation lengths for the spherical wave case are larger than the plane-wave correlation lengths, which are comparable to the correlation lengths of the refractive index.

VIII. ACKNOWLEDGMENT

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APPENDIX A

A Mean-Value Theorem

We show that any solution of

$$[\nabla^2 + k^2]u(r) = -\delta(r) \tag{119}$$

satisfies the mean-value relation

$$\frac{1}{4\pi R^2}\int dS\, u(r + \rho) = \begin{cases} u(r)\dfrac{\sin kR}{kR}, & 0 < R < |r|, \\[2ex] u(r)\dfrac{\sin kR}{kR} + \dfrac{\sin k(|r| - R)}{4\pi |r| kR}, & R > |r|, \end{cases} \tag{120}$$

where the integration is over the surface of the sphere $\{\rho: |\rho| = R\}$. In particular, when $u(r)$ is of the form (16), then (120) becomes (19).

Introduce a function $\psi(r)$ that satisfies

$$[\nabla^2 + k^2]\psi(r) = -\delta(r - r_o). \tag{121}$$

Then (119) and (121) imply that

$$\nabla\cdot(\psi\nabla u - u\nabla\psi) = u\delta(r - r_o) - \psi\delta(r). \tag{122}$$

For the sphere $\{r: |r - r_o| = R\}$ with the outward unit normal $\hat{\rho} = (r - r_o)/|r - r_o|$, the divergence theorem yields

$$\int dS\,(\psi\hat{\rho}\cdot\nabla u - u\hat{\rho}\cdot\nabla\psi) = \begin{cases} u(r_o), & 0 < R < |r_o|, \\ u(r_o) - \psi(0), & R > |r_o|. \end{cases} \tag{123}$$

We choose ψ to be a linear combination of

$$\frac{\exp(\pm ik\,|\,r - r_o\,|)}{4\pi\,|\,r - r_o\,|} \tag{124}$$

such that ψ is zero on the surface of the sphere and satisfies (121). This choice is

$$\psi(r) = -\frac{\sin k(|\,r - r_o\,| - R)}{4\pi\,|\,r - r_o\,|\sin kR}. \tag{125}$$

The radial component of the gradient of (125) evaluated on the surface of the sphere is

$$\hat{\rho}\cdot\nabla\psi = -\frac{kR}{4\pi R^2 \sin kR}. \tag{126}$$

Then (125) and (126) in conjunction with (123) yield (120).

APPENDIX B

An Improved Far-Field Condition

The kernel (41) used in the integral (40), yielding the covariance, is

$$G(r, r')G^*(\rho, \rho') = \frac{\exp[ik_o(|\,r - r'\,| - |\,\rho - \rho'\,|)]}{(4\pi)^2\,|\,r - r'\,|\,|\,\rho - \rho'\,|}. \tag{127}$$

With the definitions and inverse relations (36) to (39),

$$\begin{aligned} \eta &= \frac{r + \rho}{2}, & \xi &= r - \rho, \\ y &= \frac{r' + \rho'}{2}, & x &= r' - \rho', \end{aligned} \tag{128}$$

$$\begin{aligned} r &= \eta + \frac{\xi}{2} & \rho &= \eta - \frac{\xi}{2}, \\ r' &= y + \frac{x}{2}, & \rho' &= y - \frac{x}{2} \end{aligned} \tag{129}$$

the kernel is

$$\frac{\exp[ik_o(|\,\eta - y + \frac{1}{2}(\xi - x)| - |\,\eta - y - \frac{1}{2}(\xi - x)|)]}{(4\pi)^2\,|\,\eta - y + \frac{1}{2}(\xi - x)|\,|\,\eta - y - \frac{1}{2}(\xi - x)|}. \tag{130}$$

The far-field (Fraunhofer) approximation arises from the series

expansions (42)

$$
\begin{aligned}
|\eta - y + \tfrac{1}{2}(\xi - x)| &= |\eta - y| + \frac{1}{2}\frac{\eta - y}{|\eta - y|}\cdot(\xi - x) + \cdots, \\
|\eta - y - \tfrac{1}{2}(\xi - x)| &= |\eta - y| - \frac{1}{2}\frac{\eta - y}{|\eta - y|}\cdot(\xi - x) + \cdots,
\end{aligned}
\tag{131}
$$

and the approximation kernel (44) is

$$
\frac{\exp ik_o\left[\dfrac{\eta - y}{|\eta - y|}\cdot(\xi - x)\right]}{(4\pi)^2 |\eta - y|^2}. \tag{132}
$$

A usual condition for the validity of a far-field approximation is

$$
|\eta - y| \gg \frac{L_o^2}{\lambda}, \tag{133}
$$

where L_o is an outer scale of the scatering medium. This condition is relative to approximation of the field. But relative to approximation of the covariance of the field, the condition of validity is

$$
|\eta - y| \gg L_o\left(\frac{L_o}{\lambda}\right)^{\frac{1}{3}}. \tag{134}
$$

In the case $L_o \gg \lambda$, condition (134) is considerably less restrictive than condition (133). The reason for this improved state of affairs is that, rather than approximating Green's function G, we are approximating the kernal GG^*. In the computation of the phase of this kernel with expansion (131), there is cancellation of terms that ordinarily remain when computing the phase of G itself. Overhounding the effect of all neglected terms, not just the first one, leads to condition (134).

Our task is to approximate the phase in (130), namely, the argument of the exponential. We put

$$
Y = \eta - y, \qquad X = \xi - x, \tag{135}
$$

so that

$$
\begin{aligned}
|r - r'| &= |Y + \tfrac{1}{2}X|, \\
|\rho - \rho'| &= |Y - \tfrac{1}{2}X|.
\end{aligned}
\tag{136}
$$

Then, we ohserve

$$
|Y \pm \tfrac{1}{2}| = |Y|\left(1 \pm \frac{Y\cdot X}{Y^2} + \frac{X^2}{4Y^2}\right)^{\frac{1}{2}}. \tag{137}
$$

Next we put

$$\alpha = \frac{Y \cdot X}{Y^2}, \qquad \beta = \frac{X^2}{4Y^2}, \tag{138}$$

so that

$$\begin{aligned} | r - r' | &= | Y | (1 + \alpha + \beta)^{\frac{1}{2}}, \\ | \rho - \rho' | &= | Y | (1 - \alpha + \beta)^{\frac{1}{2}}. \end{aligned} \tag{139}$$

The next step is to assume $| \pm \alpha + \beta | < 1$ and expand $| r - r' |$ and $| \rho - \rho' |$ with a binomial series. Then,

$$(1 \pm \alpha + \beta)^{\frac{1}{2}} = 1 + \tfrac{1}{2}(\pm\alpha + \beta) - \frac{1 \cdot 1}{2 \cdot 4}(\pm\alpha + \beta)^2$$
$$+ \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}(\pm\alpha + \beta)^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}(\pm\alpha + \beta)^4 + \cdots . \tag{140}$$

In the above expression, only terms with differing signs contributed to the difference $| r - r' | - | \rho - \rho' |$. Thus,

$$(1 + \alpha + \beta)^{\frac{1}{2}} - (1 - \alpha + \beta)^{\frac{1}{2}}$$
$$= \alpha - 2\frac{1 \cdot 1}{2 \cdot 4} 2\alpha\beta + 2\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\alpha^3 + 2\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} 3\alpha\beta^2 + R, \tag{141}$$

where the remainder R has the series expansion

$$R = -\frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}[(\alpha + \beta)^4 - (-\alpha + \beta)^4]$$
$$+ \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}[(\alpha + \beta)^5 - (-\alpha + \beta)^5] - \cdots . \tag{142}$$

The series is readily "majorized," with the result

$$| R | < 2 \left| \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}(| \alpha | + | \beta |)^4 \right.$$
$$\left. + \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}(| \alpha | + | \beta |)^5 + \cdots \right|, \tag{143}$$

or, with $\gamma = | \alpha | + | \beta |$ and $0 < \gamma < 1$,

$$\frac{| R |}{2} < \frac{5}{128}\gamma^4\left[1 + \frac{7}{10}\gamma + \frac{7 \cdot 9}{10 \cdot 12}\gamma^2 + \cdots\right]. \tag{144}$$

But the series in brackets is overbounded by

$$\frac{1}{1-\gamma} = 1 + \gamma + \gamma^2 + \cdots . \tag{145}$$

Thus, $\gamma < \frac{1}{2}$ implies

$$| R | < \frac{5}{32}\gamma^4 = \frac{5}{32}(| \alpha | + | \beta |)^4. \tag{146}$$

The above calculations show that

$$| r - r' | - | \rho - \rho' | = | Y | \left[\frac{Y \cdot X}{Y^2} - \frac{1}{8}\frac{Y \cdot X}{Y^2}\frac{X^2}{Y^2} + \frac{1}{8}\left(\frac{Y \cdot X}{Y^2}\right)^3 + \frac{3}{128}\frac{Y \cdot X}{Y^2}\left(\frac{X^2}{Y^2}\right)^2 + R\right], \tag{147}$$

where

$$| R | < \frac{5}{32}\left(\left|\frac{Y \cdot X}{Y^2}\right| + \frac{X^2}{4Y^2}\right)^4. \tag{148}$$

The conditions

$$\left| \pm \frac{Y \cdot X}{Y^2} + \frac{X^2}{4Y^2} \right| < 1 \tag{149}$$

and

$$\left|\frac{Y \cdot X}{Y^2}\right| + \frac{X^2}{4Y^2} < \frac{1}{2} \tag{150}$$

have been imposed. Since condition (150) implies (149) it is clear that both are met when

$$\frac{| X |}{| Y |} < \frac{1}{3}. \tag{151}$$

The far-field approximation of the kernel employs the leading term of (147); that is to say, the Fraunhofer phase is

$$k_o \frac{Y}{| Y |} \cdot X = \frac{2\pi}{\lambda} \frac{\eta - y}{| \eta - y |} \cdot (\xi - x). \tag{152}$$

The phase error is then

$$-k_o\left\{\frac{| X |}{8} \cdot \frac{Y \cdot X}{| Y | | X |} \cdot \frac{X^2}{Y^2} \cdot \left[1 - \left(\frac{Y \cdot X}{| Y | | X |}\right)^2 - \frac{3}{16}\frac{X^2}{Y^2}\right] - | Y | R\right\}. \tag{153}$$

Our task now is to overbound the magnitude of the phase error. We impose $|X| \leqq L_o$, an outer scale of the refractive-index correlation function Γ. This is appropriate for zero receiver separation, $|\xi| = 0$; later, L_o could be replaced by $|\xi| + L_o$ for nonzero receiver separation. Apart from the remainder, (153) is seen to be a function of the cosine $u \cdot v$, with $u = X/|X|$ and $v = Y/|Y|$, and of the ratio $w = |X|/|Y|$. We overbound the product in (153) by a product of overbounds in which $u \cdot v$ has distinct values, and overbound the second factor by unity (unity is greater than both the largest positive value $1 - \frac{3}{16}w^2$ and the largest magnitude, $\frac{3}{16}w^2$, of negative values, since $w < 1$). It follows that the magnitude of the phase error is overbounded by

$$k_o\left(\frac{L_o^3}{8Y^2} + |Y||R|\right). \tag{154}$$

We now overbound our previous estimate of $|R|$. We assume $|Y| > L_o$, even though tighter overhounds can be obtained under $|Y| > 3L_o$ as previously assumed. Then, (148) yields

$$|R| < \frac{5}{32}\left(\frac{L_o}{|y|} + \frac{L_o^2}{4Y^2}\right)^4 < \frac{5}{32}\left(\frac{5L_o}{4|Y|}\right)^4, \tag{155}$$

or

$$|R| < \frac{1}{10}\left(\frac{25}{16}\right)^3\left(\frac{L_o}{|Y|}\right)^4 < 0.382\left(\frac{L_o}{|Y|}\right)^4. \tag{156}$$

We use

$$|R| < \frac{2}{5}\left(\frac{L_o}{|Y|}\right)^4. \tag{157}$$

The above phase-error bound (154) is less than

$$k_o \frac{L_o^3}{8Y^2}\left(1 + \frac{16}{5}\frac{L_o}{|Y|}\right), \tag{158}$$

which is less then

$$\frac{k_o L_o^3}{8Y^2} \cdot \frac{21}{5} < \frac{5}{8}\frac{k_o L_o^3}{Y^2}. \tag{159}$$

All of the above calculations required $|Y| > L_o$. With the stronger condition $|Y| > 4L_o$, the next -to-last calculation becomes

$$k_o \frac{L_o^3}{8Y^2}\left(1 + \frac{16}{5}\frac{L_o}{|Y|}\right) < \frac{k_o L_o^3}{8Y^2}\frac{9}{5}. \tag{160}$$

This overbound is less than

$$\frac{k_o L_o^3}{4Y^2}. \tag{161}$$

Suppose we impose the condition that the phase error be less than $\pi/32$ and we ask what value of $| Y |$, the scattering range, is required. The last overbound yields

$$| \eta - y | = | Y | > 4\left(\frac{L_o}{\lambda}\right)^{\frac{1}{2}} L_o . \tag{162}$$

This is a less restrictive condition than one specifying the far-field range to be much greater than L_o^2/λ. When $L_o \gg \lambda$, it is a considerably weaker condition. On the other hand, with $L_o > \lambda$ (but now $L_o \approx \lambda$), the condition (162) still implies the assumption $| Y | < 4L_o$ under which it was obtained. When $L_o < \lambda$, the results are valid but vacuous, since the pertinent condition is then $| Y | > 4L_o$.

We turn to the approximation of the coherent-field function (46),

$$u\left(y + \frac{x}{2}\right)u_c^*\left(y - \frac{x}{2}\right) = \frac{\exp\left[i\left(k\left|y + \frac{x}{2}\right| - k^*\left|y - \frac{x}{2}\right|\right)\right]}{(4\pi)^2\left|y + \frac{x}{2}\right|\left|y - \frac{x}{2}\right|}, \tag{163}$$

in which k is a weak function of position. With the identification $y = Y, x = X$, the previous analysis is applicable. The approximation (48) to (50) is

$$\frac{\exp(-2 \operatorname{Im} k \mid y \mid) \exp\left[i(\operatorname{Re} k) \frac{y}{| y |} \cdot x\right]}{(4\pi)^2 \mid y \mid^2}, \tag{164}$$

with a phase error less than

$$\frac{\operatorname{Re} k L_o^3}{4 \mid y \mid^2}. \tag{165}$$

In view of the interchangeability of k and k_o, the phase error (165) is comparable to

$$\frac{k_o L_o^3}{4 \mid y \mid^2}. \tag{166}$$

Then, a phase error in (164) less than $\pi/32$ requires that the source-to-scattering distance $| y |$ satisfy

$$| y | > 4L_o(L_o/\lambda)^{\frac{1}{2}}. \tag{167}$$

Our calculation is similar to one by Lahti and Ishimaru, but the calculation (and result) is simpler and the final conditions are less restrictive.[10] Simplicity is obtained in part because we use variables with symmetric form, (128) and (129), and we overbound a quartic remainder rather tban modify a cubic remainder. Our quartic-remainder overhound also yields less restrictive conditions, say $|Y| > 4L_o(L_o/\lambda)^{\frac{1}{2}}$ implying a phase error less than $\pi/32$ in comparison with $|Y| > 7L_o(L_o/\lambda)^{\frac{1}{2}}$ yielding an error less than $\pi/10$.

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